Lec 23 : Poisson distribution

$X \sim \text{Poisson}(\lambda) \quad x \in \{0, 1, \cdots\}$

$p_k = P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!} \quad \left[\sum_{k=0}^{\infty} \dfrac{e^{-\lambda}\lambda^k}{k!} = 1 \right] \to \sum_{k=0}^{\infty} p_k$

"# of people arriving at a store in a one hour period."

$E[X] = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} \dfrac{k e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \dfrac{\lambda^k}{(k-1)!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \dfrac{\lambda^{k-1}}{(k-1)!}$

value

$= e^{-\lambda}\lambda \sum_{t=0}^{\infty} \dfrac{\lambda^t}{t!} e^{\lambda} = \lambda$

Can compute $E[X(X-1)] = \lambda^2$ and thus

$Var(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

$(E[X] = Var(X) = \lambda)$

Prop: If X, Y are Poisson (μ), Poisson (ν) are are independent then $X+Y \sim \text{Poisson}(\mu+\nu)$

$E[e^{tX}] = M(t) = \sum_{k=0}^{\infty} \underbrace{e^{tk}}_{\text{value}} \dfrac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \dfrac{(e^t\lambda)^k}{k!}$

$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{\mu(e^t-1)} e^{\nu(e^t-1)}$

Quiz tonight.

Practice exams from a previous class.

→ Branching process / Birth + death process

→ Theory of martingales

→ May 5th 8:30-11:30 (11:25 - upload our exams)

→ Personal zoom.

$e^{-\lambda} \sum_{k=0}^{\infty} \dfrac{\lambda^k}{k!} = 1 \iff \sum_{k=0}^{\infty} \dfrac{\lambda^k}{k!} = e^{\lambda}$

λ is the avg. value of Poisson (λ) & represents the avg # of customers coming in.

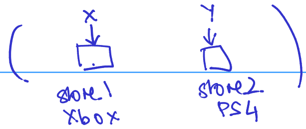
$E[X^2 - X] = \lambda^2 \Rightarrow E[X^2] = \lambda^2 + \lambda$

$M_{X+Y}(t) = M_X(t) M_Y(t)$ if X and Y independent

$M_\lambda(t) = e^{\lambda(e^t - 1)}$

$$= e^{(\mu+\nu)(e^t-1)}$$ → This MGF of a Poisson.

$\Rightarrow X+Y \sim \text{Poisson}(\mu+\nu)$.



<u>Poisson Process</u> suppose we have 2 customers arriving at a store on an <u>average</u> in 1 hour.

Then <u>$X \sim \text{Poisson}(2)$</u> could be a good model for the random # of customers arriving between 9-10am

2 customers 1 hour

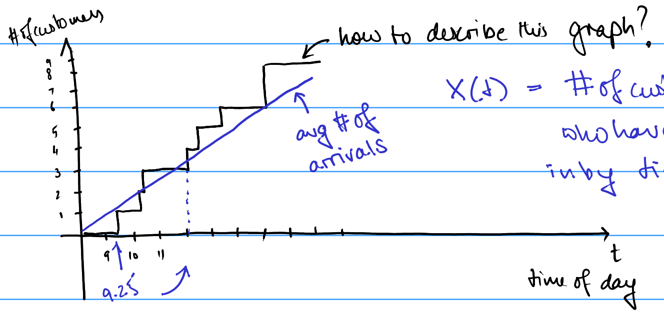
$2 \cdot 2.5 = 5$ customers in 2.5 hours on avg.

Y = # of people coming in between 9:30 and 12am?

$Y \sim \text{Poisson}(5)$

How to model the # of people coming in between 9:30 and 12am?

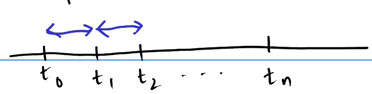
$Y = X(12) - X(9{:}30)$



$X(t)$ = # of customers who have come in by time t.

<u>Def</u>: Poisson process of rate $\lambda > 0$

1)



$X(t_1) - X(t_0), \cdots, X(t_n) - X(t_{n-1})$ are independent.

# of arrivals in (t_0, t_1)

λ = # of people coming in on average in one hour

→ Arrivals in disjoint time intervals are INDEPENDENT.

2) $\underbrace{X(s+t)-X(s)}_{\#\text{ of arrivals in } (s,t+s)} \sim \text{Poisson}(\lambda t)$

3) $X(0) = 0.$

$E[X(t)] = \lambda t \quad Var(X(t)) = \lambda t \qquad X(t) \sim \text{Poisson}(\lambda t)$

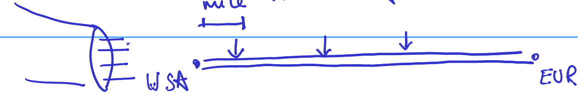
Ex: Defects along an undersea cable appear according to a Poisson process at rate $\lambda = 0.1$ per mile.

1) Prob of no defects in 1st 2 miles?

2) Given no defects in 1st 2 miles what's the prob of no defects between mile points 2 and 3.

$\lambda = 2 \qquad Y[\overset{s}{9{:}30}, \overset{s+t}{12}] \sim \text{Poisson}(2 \cdot 2.5)$

mile $\quad \lambda = 0.1$ defect per mile



Poisson process with rate λ. $X(t)$ ← length

$X(t) = \#$ of defects by mile t.

$P(X(2) = 0)$ ← no defects

$$= \frac{e^{-0.2}(0.2)^0}{0!} = e^{-0.2}\frac{1}{1}$$

$X(2) \sim \text{Poisson}(\lambda \cdot 2) = \text{Poisson}(0.2)$

2) $A = \{X(2) = 0\} \quad B = \{X(3) - X(2) = 0\}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(X(3)-X(2)=0,\ X(2)=0)}{P(X(2)=0)}$$

$$= \frac{P(X(3)=0)}{P(X(2)=0)} = \frac{e^{-0.3}}{e^{-0.2}}$$

Ex: Store arrivals are Poisson with $\lambda = 4$ per hour.

If store opens at 9:00am, find

$P(X(9{:}30) = 1$ and $X(11{:}30) = 5)$

(There are textbook examples)

$$= P(\underbrace{X(9{:}30) = 1}_{\sim \text{Poisson}(4 \cdot 0.5)})\, P(\underbrace{X(11{:}30) - X(9{:}30) = 4}_{\text{Poisson}(4 \cdot 2)})$$

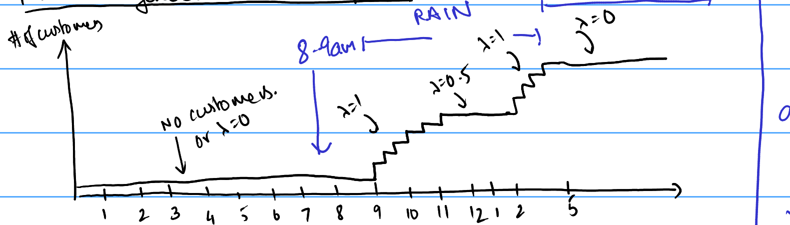
$$= e^{-2} \frac{(2)^1}{1!} \, e^{-8} \frac{(8)^4}{4!}$$

$$P(X(9{:}30) = 1 \cap X(11{:}30) = 5)$$

$$\cancel{=} P(X(9{:}30) = 1)\, P(X(11{:}30) = 5)$$

$$= P(\underbrace{X(9{:}30) = 1}_{X(9{:}30) - X(0)} \cap \underbrace{X(11{:}30) - X(9{:}30) = 4}_{\text{independent increment.}})$$

Nonhomogeneous Poisson process



λ is changing with t and represents the avg # of customers arriving.

$\lambda(t)$.

How to accomodate a changing λ?

$P(\underbrace{X(t+h)-X(t)}_{\#\text{ of arrivals}}=1)$ = "prob of an arrival in $[t,t+h)$" (h is small)

Infinitesimal arrival rate.

Poisson(λh)

$$= \frac{(\lambda h)^1 e^{-\lambda h}}{1!} \approx \lambda h\left(1-\lambda h+\frac{\lambda^2 h^2}{2!}+\dots\right)$$

$$\lambda h - \underbrace{\left(\lambda^2 h^2 + \frac{\lambda^3 h^3}{2!} \dots\right)}_{= A}$$

these are very small for $h \downarrow 0$.

$$\approx \lambda h + \boxed{o(h)}$$

Recall $A = o(h)$ if $\lim_{h\downarrow 0} \frac{A}{h} = 0$.

$$\Rightarrow \lim_{h\downarrow 0} \frac{P(X(t+h)-X(t)=1)}{h} = \lim_{h\downarrow 0} \frac{\lambda h + o(h)}{h} = \lambda$$

THUS λ is an "instantaneous rate of arrival"

We could make λ vary with time and define a new Poisson process such that $\lambda(t)$ is the "instantaneous rate of arrival"

1) $X(t+s) - X(t) \sim \text{Poisson}\left(\int_t^{t+s} \lambda(u)\,du\right)$

2) $t_0 \quad t_1 \quad t_2 \quad \cdots \quad t_n$

$\underset{\text{\# of arrivals in } (t_0,t_1)}{X(t_1)-X(t_0)}, \dots, X(t_n)-X(t_{n-1})$ are independent.] independent increments

3) $X(0) = 0$.

Then if $\lambda(t) = \lambda$ (constant) this is just the regular Poisson process.

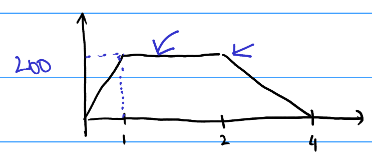
$$\overset{[t, t+s)}{X(t+s) - X(t)} \sim \text{Poisson}\left(\int_t^{t+s} \lambda\, du\right) = \text{Poisson}(\lambda s)$$

Ex: Suppose you are a hospital preparing for Covid 19. Your growth model predicts an average rate of arrival at the hospital that varies with time as follows.

$$\lambda(t) = \begin{cases} 200t & 0 \le t < 1 \\ 200 & 1 \le t < 2 \\ 400 - 100t & 2 \le t < 4 \end{cases}$$

t is measured in weeks

where t is measured in weeks.



What is the probability that 400 patients arrive in the 1st two weeks and 400 more arrive in the 2nd two weeks?

$$P\big(X(2) = 400,\ \overbrace{X(4) - X(2)}^{\text{Poisson}(200)} = 400\big)$$

$$\int_2^4 \lambda(u)\,du = \frac{1}{2}\cdot 200 \cdot 2 = 200$$

$$\underbrace{\int_0^2 \lambda(u)\,du}_{\text{Poisson}(300)} = \frac{1}{2}\cdot 200 \cdot 1 + 1 \cdot 200 = 300$$

$$= \frac{e^{-300}(300)^{400}}{400!} \cdot \frac{e^{-200}(200)^{400}}{400!}$$

$$P(X(2)-X(0)=400 \cap X(4)-X(2)=400)$$

$$= P(\underbrace{X(2)-X(0)}_{\text{Poisson}\left(\int_0^1 200t + \int_1^2 200\right)}=400)\, P(\underbrace{X(4)-X(2)}_{\text{Poisson}\left(\int_2^4 400-100t\right)}=400)$$

You can estimate these probabilities to plan for various scenarios.

Time change: let $X(t)$ be inhomogeneous Poisson with rate $\lambda(t) > 0$

big/capital λ

let $\Lambda(t) = \int_0^t \lambda(s)\,ds$. (strictly increasing)

let $Y(s) := X(t)$ where $t = \Lambda^{-1}(s)$

$Y(s) = X(\Lambda^{-1}(s)) \leftarrow$ changed time

Claim: $Y(s)$ is Poisson with rate 1.

Note $s = \Lambda(t) \Rightarrow \boxed{\frac{ds}{dt} = \Lambda'(t) = \lambda(t)}$ (FTC)

$\Rightarrow \Delta s \approx \lambda(t)\,\Delta t$

$P(Y(s+\Delta s) - Y(s) = 1) = P(\underbrace{X(t+\Delta t) - X(t)}_{\text{Poisson}\left(\int_t^{t+\Delta t} \lambda(u)du\right)} = 1)$

$\approx \int_t^{t+\Delta t} \lambda(u)du \approx \lambda(t)\Delta t \approx \Delta s \quad (\Delta t \downarrow 0)$

$\Rightarrow$ Y is a Poisson process with rate 1.

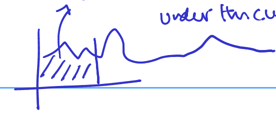
(See alternative characterization of Poisson process)

In particular if $\lambda(t) = \lambda$ $\quad \Lambda(t) = \lambda t$

$Y(s) = X(s/\lambda)$ has rate 1.

whereas $X(t)$ has rate λ.

$\Lambda(t)$ is the area under the curve

$\lambda(t) =$ 

$X(t) \to Y(t)$

constant λ Poisson process by a change of variable.

$\frac{d}{dt}\int_0^t \lambda(s)ds = \lambda(t)$ FTC

$\lim_{\Delta s \downarrow 0} \frac{P(Y(s+\Delta s) - Y(s))}{\Delta s} = \lambda$

Poisson $(\lambda(t)\Delta t + o(\Delta t))$

$P(Y(s+\Delta s) - Y(s)) \approx \Delta s$

$X(t) \sim$ Poisson (λ)

$Y(s) = X(t) \qquad t = \Lambda^{-1}(s) = \frac{s}{\lambda}$

$Y(s) \sim$ Poisson(1)

$Y(s) = X(s/\lambda)$

Cox-Miller. (I think its the same Cox)

Cox Processes

We've gone from constant λ to time varying $\lambda(t)$. → deterministic function.

$$\lambda(t) = \begin{cases} 10 & 9 < t < 11 \\ 5 & 11 < t < 4 \\ 1 & 4 < t < 5 \\ 0 & t > 5 \end{cases}$$

What if $\lambda(t)$ is itself random?

This is called a doubly stochastic process. Nowadays called a Cox Process.

Ex: Suppose customers arrive at a store and the # of customers arriving a particular day depends on the weather:

→ $X(t)$ is Poisson rate λ

$Y(t) = X(t/\lambda)$ is Poisson rate 1

$X(t) = Y(\lambda t)$

$$W = \begin{cases} \text{Sunny} & P = 1/10 \\ \text{Rain} & P = 5/10 \\ \text{Cloudy} & P = 4/10 \end{cases}$$

$$\lambda(W) = \begin{cases} 20 & W = S \\ 2 & W = R \\ 5 & W = \text{Cloudy} \end{cases}$$

$Y(t)$ = # of customers arriving on a particular day by time t.

Let $X(t)$ be a rate 1 Poisson process

Then $Y(t) = X(\lambda(W)\, t)$

weather

$$P(Y(t)=k) \overset{?}{=} P(Y(t)=k,\ W=\text{Sunny}) + P(Y(t)=k,\ W=\text{Rainy}) + P(Y(t)=k,\ W=\text{Cloudy})$$

$$= \overbrace{P(Y(t)=k \mid W=S)}^{\text{Poisson}(20t)} P(W=S) + \overbrace{P(Y(t)=k \mid W=R)}^{\text{Poisson}(2t)} P(W=R)$$

$$P(\underbrace{Y(t)=k}_{P(5t)} \mid W=C)\, P(W=C)$$

$$=$$

$i \leftrightarrow j$ if $\exists n$ st $P_{ij}^{n} > 0$ $\exists m$ $P_{ji}^{m} > 0$

$1 \leftrightarrow 2$ $P_{12}^{1} = \frac{1}{2} > 0$ $P_{21}^{1} = \frac{1}{2} > 0$ $1 \leftrightarrow 3$

Lec24: Law of rare events (5.3)

Lesson: Binomial $(n, \frac{\lambda}{n})$ approximates Poisson (λ).

$\bullet$ $\bullet$ . . . . . $\bullet$ ← trials. (flip a coin)

1 . . . . . N

$X_{n,p}$ = # of successes in N trials.

$$P(X_{n,p} = k) = \frac{N!}{k!\,(N-k)!} p^k (1-p)^{N-k}$$

Let $Np = \lambda$

$$= \frac{N!}{k!\,(N-k)!} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

$$\xrightarrow{N \to \infty} \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{(poisson pmf)}$$

$\left(p = \frac{\lambda}{N}\right)$ "Prob of success is proportn to $\frac{1}{\text{\# of trials}}$"

★ HW.

In other words

When the probability of success

$\propto \frac{1}{N} = \frac{1}{\text{\# of trials}}$ (large # of trials)

"rare" (not really)

Then Binomial $\approx$ Poisson

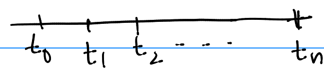
Thm: Let $X \sim Bin(n,p)$ $Y \sim Poisson(\lambda)$

$$\left| P(X=k) - P(Y=k) \right| \leq np^2$$

So if $p = \frac{\lambda}{n}$ $\quad np^2 = \frac{\lambda^2}{n} \longrightarrow 0$ as $n \to \infty$.

New Postulates for a Poisson Process

Let $N(t)$ be a stochastic process $N(t)$ $\in \{0,1,\ldots\}$

1) 

] Independent increments.

$N(t_1) - N(t_0), \ldots, N(t_n) - N(t_{n-1})$ indep.

2) $P(N(t+h) - N(t) = k) = f(h)$
(that is, independent of t). Time homogeneity.

3) $P(N(t+h) - N(t) \geq 1) = \lambda h + o(h)$
(Arrivals in any given small interval are "rare")

$n = 1000$ $\quad p = 0.01$

$np^2 = 10^3 (10^{-2})^2 = 10^{-1}$

$$\left| \binom{n}{k} p^k (1-p)^{n-k} - \frac{e^{-\lambda} \lambda^k}{k!} \right| \leq 0.1$$

$\lambda = np = 10^{-2} 10^3 = 10$

1) Independent increments $X(t_2) - X(t_1)$, $X(t_1) - X(t_0)$
$t_0 < t_1 < t_2$

2) $X(t) = 0$

3) $X(t) - X(s) \sim Poisson(\lambda(t-s))$

Prob(k arrivals in the time interval $[t, t+h)$)
$= f(h)$

P(more than 1 arrival in $[t, t+h)$)
$= \lambda h + o(h)$ $\quad h \downarrow 0$

Previously we had computed

$P(X(t+h)-X(t)=1) = \lambda h + o(h)$ (★1)

Poisson process.

$A = o(h)$ if $\lim_{h \to 0} \frac{A}{h} = 0$

4) $P(N(t+h)-N(t) \geq 2) = o(h)$ → $P(N(t+h)-N(t) \geq 2) = o(h)$

2 or more arrivals extremely unlikely.

4) and 3) ⇒ (★1)

★ HW.

Thm: A stochastic process satisfying above assumptions is a Poisson process.

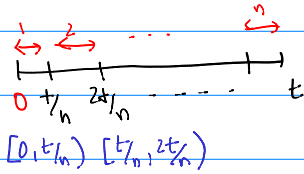
(Postulates 1–4 ⇒ previous 3 defining postulates of Poisson)

The idea is as follows: need to show

$$P(N(t)=k) = \frac{(\lambda t)^k e^{-\lambda}}{k!}$$

2) ⇒ $N(t+h)-N(t) \sim \text{Poisson}(\lambda t)$

Use Binomial approximation theorem.



divide it up into little intervals.

$$P\left(N\left(\frac{kt}{n}\right) - N\left(\frac{(k-1)t}{n}\right) = 1\right) = \lambda \underbrace{\left(\frac{t}{n}\right)}_{h} + o\left(\frac{t}{n}\right)$$ ← 3), 4)

→ "probability of success in any of the little intervals"

Using assumption 1) (independent increments)

$P(N(t)=k) = P(\text{exactly } k \text{ of the intervals have one arrival}) = P(S_n = k) \approx \binom{n}{k} p^k (1-p)^{n-k}$ (★1a)

$$\to \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

But if S_n = # of successful intervals.

$$\textcircled{\star 1a} = P(S_n = k)$$

where $S_n \sim \text{Bin}\Big(n, \underbrace{\frac{\lambda t}{n} + o(t/n)}_{p}\Big)$

Let $Y = \text{Poisson}\Big(\underbrace{\lambda t + n\,o(t/n)}_{\mu}\Big)$

By our theorem

$$\Big| P(S_n = k) - \underbrace{\frac{e^{-\mu}\mu^k}{k!}}_{\text{pmf of Poisson}} \Big| \le n p^2 \quad \textcircled{\star 2}$$

$$\mu = \lambda t + \underbrace{n\,o(t/n)}_{\downarrow 0}$$

where

p = Prob of an arrival in each of these intervals

$$= \frac{\lambda t}{n} + o\left(\frac{t}{n}\right)$$

$$\textcircled{\star 2} \le n\left(\frac{\lambda t}{n} + o\left(\frac{t}{n}\right)\right)^2$$

$$= n\Big(\underbrace{\frac{\lambda^2 t^2}{n^2}}_{\downarrow 0} + 2\lambda t\, o\left(\frac{t}{n}\right) + \left(o(t/n)\right)^2\Big)$$

$$\overset{\nearrow 0}{\frac{\lambda^2 t^2}{n}} + 2\lambda t \underbrace{\boxed{n\,o(t/n)}}_{\downarrow 0} + n\,o(t/n)^2$$

By definition

$$\frac{o(t/n)}{t/n} = \frac{n\, o(t/n)}{t} \xrightarrow{n\to\infty} 0 \qquad \frac{o(h)}{h} \xrightarrow[h\downarrow 0]{} 0$$

$\Rightarrow\ o(t/n) \longrightarrow 0$

$$n\left[o\left(t/n\right)\right]^2 = \frac{o\left(t/n\right)^2}{t/n}\, t$$

$$= t\,\frac{o(h)^2}{h} = t\, \cancel{o(h)}^{\nearrow 0}\, \frac{\cancel{o(h)}}{h}^{\nearrow 0}$$

But

$$\left| P(S_n = k) - \frac{e^{-\lambda t}(\lambda t)^k}{k!} \right|$$

$\leftarrow$ adding and subtracting $\frac{e^{-\mu}\mu^k}{k!}$

$$\leq \left| P(S_n = k) - \frac{e^{-\mu}(\mu)^k}{k!} \right| \to 0 \text{ by previous calculation}$$

$$+ \left| \frac{e^{-\mu}\mu^k}{k!} - \frac{e^{-\lambda t}(\lambda t)^k}{k!} \right| \to 0$$

$$\frac{e^{-(\lambda t + \overbrace{n\, o(t/n)}^{\to 0})}}{k!}\left(\lambda t + n\, o(t/n)\right)^k$$

using Δ inequality

So we have seen 2 ways of looking at the poisson process

1) By postulating indep. increments and requiring Poisson distribution

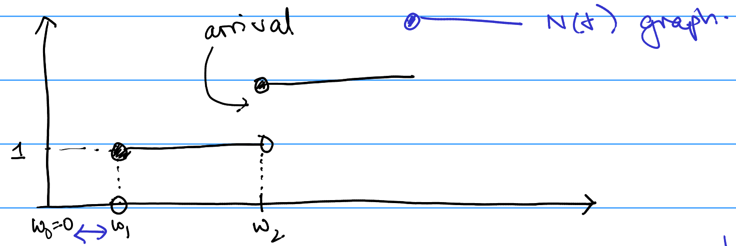
2) By postulating indep. increments and DERIVING Poisson distribution using a hypothesis stating that

$P(N(t+h)-N(h)=1) = \lambda h + o(h)$ - arrival in a small interval is rare

$P(\underbrace{N(t+h)}-N(h)\geq 2) = o(h)$

No simultaneous arrivals.

5.3 (Exponential and Gamma distribution)



Time of occurrence of the n^{th} arrival

$= W_n$ (Waiting Time)

$S_n = W_n - W_{n-1}$ (Sojourn Time)

Thm: $W_n \sim$ Gamma (λ, n)

$$f_{W_n} = \begin{cases} \dfrac{\lambda^n t^{n-1}}{\Gamma(n)} e^{-\lambda t} & t>0 \quad n=1,2,\dots \\ 0 & t \le 0 \end{cases}$$

$\Gamma(n) =$ Gamma function $= (n-1)!$

$$f_{W_1} = \begin{cases} \lambda e^{-\lambda t} & t>0 \\ 0 & t \le 0 \end{cases}$$

$S_n =$ "interarrival time"

W_1 = Exponential distribution

$W_2 - W_1 \sim$ distribution?

$$\underbrace{W_2}_{Gamma(2,\lambda)} = \underbrace{W_2 - W_1}_{Exp(\lambda)} \overset{indep}{+} \underbrace{W_1}_{Exp(\lambda)}$$

Recall: $\{X_k\}$ iid $X_k \sim Exp(\lambda)$

$X_1 + X_2 + \cdots + X_k \sim$ convolution

$X_1 + X_2 + \cdots + X_k \sim Gamma(k, \lambda)$

Thm: Sojourn times (interarrival) $\{W_n - W_{n-1}\}_{n=1}^{\infty}$ are iid exponentials. with rate λ

Pf: $F_{W_n}(t)$ (CDF) (show that it is the CDF of the gamma dist.)

$= P(W_n \leq t) = P($n arrivals have happened before time t$)$

$= P(\underset{\text{\# arrivals at time t}}{X(t)} \geq n)$

$X(t) \sim Poisson(\lambda t)$

$= P(Poisson(\lambda t) \geq n)$

$$= \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$\text{Then } \overbrace{f_{w_n}(t)}^{\text{pdf of } w_n(t) \text{ by differentiating } F_{w_n}(t)} = \sum_{k=n}^{\infty} \frac{d}{dt}\left(e^{-\lambda t}\frac{(\lambda t)^k}{k!}\right)$$

$$= \sum_{k=n}^{\infty}\left(-\lambda e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \frac{\cancel{k}\lambda^k t^{k-1} e^{-\lambda t}}{\underset{(k-1)!}{\cancel{k!}}}\right)$$

$$= e^{-\lambda t}\lambda\left[\sum_{k=n}^{\infty} -\frac{(\lambda t)^k}{k!} + \sum_{k=n}^{\infty}\frac{(\lambda t)^{k-1}}{(k-1)!}\right]$$

$j = k-1$ $\quad \displaystyle\sum_{j=n-1}^{\infty}\frac{(\lambda t)^j}{j!}$

$$= e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}\lambda = \frac{e^{-\lambda t}\lambda^n t^{n-1}}{(n-1)!} \quad \blacksquare$$

Thm: $\overbrace{S_0, \dots, S_{n-1}}^{\text{interarrival}}$ are iid $\text{Exp}(\lambda)$

Pf: To show

$$\underbrace{f_{S_0 \dots S_{n-1}}}_{\text{joint pdf}}(t_0, \dots, t_{n-1}) = \underbrace{\lambda e^{-\lambda t_0} \cdots \lambda e^{-\lambda t_n}}_{\text{product of Exp densities.}}$$

For $n=2$

Will compute $P(S_0 \le t_0, S_1 \le t_1)$

and show $= \underbrace{(1-e^{-\lambda t_0})}_{\text{cdf of Exp}(\lambda)}\underbrace{(1-e^{\lambda t_1})}_{\text{cdf of Exp}(\lambda)}$ (Product form implies independendance)

$$P(S_0 \le t_0, S_1 \le t_1) = \int_0^{t_0} P(S_1 \le t_1 \mid S_0 = u)\, f_{S_0}(u)\,du$$

$$= \int_0^{t_0} P(X(u+t_1) \geq 2 \mid X(u) = 1)\, f_{S_0}(u)\,du$$

$$= \int_0^{t_0} \frac{P(X(u+t_1) - X(u) \geq 1, X(u) = 1)}{P(X(u)=1)}\, f_{S_0}(u)\,du$$

$$= \int_0^{t_0} \frac{P(X(u+t_1) - X(u) \geq 1)\, P(\cancel{X(u)}=1)}{P(X\cancel{(u)}=1)}\, \underbrace{f_{S_0}(u)}_{\lambda e^{-\lambda u}}\,du$$

$(1 - e^{-\lambda(t_2 - u)})$

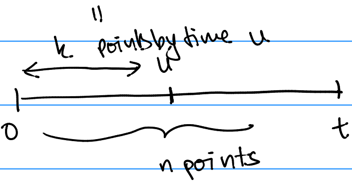
$$\int_0^{t_0} (1 - e^{-\lambda t_1})\, \lambda e^{-\lambda u}\,du$$

$$= (1 - e^{-\lambda t_0})(1 - e^{-\lambda t_1}) = P(S_0 \le t_0)\, P(S_1 \le t_1)$$

Finally we can show that the Binomial distribution can be found as well.

Thm: Let $X(t)$ be Poisson. Then $0 < u < t$

$$P(X(u) = k \mid \overbrace{X(t) = n}) = \underbrace{\binom{n}{k}\left(\frac{u}{t}\right)^k \left(1-\frac{u}{t}\right)^{n-k}}_{\text{Binomial}}$$



$$P(X(u) = k \mid X(t) = n)$$

They are not.

$$= \frac{P(X(u) = k, X(t) = n)}{P(X(t) = n)}$$

n-k more events.

$$= \frac{P(\overline{X(u) = k}, \overline{X(t) - X(u)} = \overline{n-k})}{P(X(t) = n)}$$

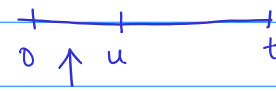
$$= \frac{e^{-\lambda u}\dfrac{(\lambda u)^k}{k!} \; \dfrac{e^{-\lambda(t-u)}(\lambda(t-u))^{n-k}}{(n-k)!}}{e^{-\lambda t}(\lambda t)^n / n!}$$

Gamma distribution

Exponential distribution

"independent exponentials"

condition on n arrivals.



What's the prob. that k arrivals happen in $[0, k)$

$$\{X(u) = k, X(t) = n\}$$

$$= \{X(u) = k, X(t) - X(u) = n\}$$

$$\frac{n!}{k!(n-k)!} \; \frac{(\lambda u)^k (\lambda(t-u))^{n-k}}{(\lambda t)^n}$$

$$= \frac{n!}{k!(n-k)!} \; \frac{u^k (t-u)^{n-k}}{t^n} = \binom{n}{k} \left(\underbrace{\frac{u}{t}}_{p}\right)^k \left(\underbrace{1-\frac{u}{n}}_{1-p}\right)^{n-k}$$

Another way to think about it:

Suppose we know there are n points in the interval $[0,t]$. $X(t)=n$

Assume that all n points are indep. uniformly distributed on $[0,t]$

Then whats the prob. that the first pt lies in $[0,u]$? $= \underbrace{\left(\frac{u}{t}\right)}_{p} = P(\text{Uniform}[0,t] \le u)$

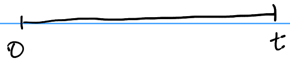
(Uniform dist)

Assuming independence:

$P(\text{k points in } [0,u] \mid X(t)=n) = P\left(\text{Binomial}\left(n, \frac{u}{t}\right) = k\right)$

$$= \binom{n}{k}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}$$

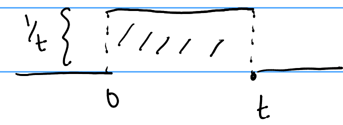
Lec25: (Uniform distribution and the Poisson process)



lets take $\{U_i\}_{i=1}^{n}$ $U_i \sim \text{Uniform}[0,t)$

iid.

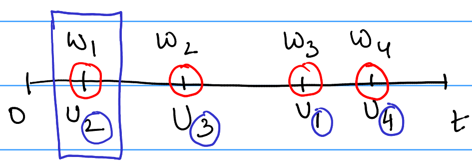
$$f_{U_i}(s) = \begin{cases} \frac{1}{t} & 0 \le s < t \\ 0 & \text{otherwise} \end{cases} \quad \text{pdf}$$



$$f_{U_1 \cdots U_n}(w_1', \dots, w_n') = f_{U_1}(w_1') \cdots f_{U_n}(w_n')$$

$$= \begin{cases} \frac{1}{t^n} & 0 \le w_i' \le t \\ 0 & \text{otherwise.} \end{cases}$$

let $W_1 < W_2 \cdots < W_n$ be the order in which $U_1, \dots, U_n$ appear on the line.



let $\sigma = (2\ 3\ 1\ 4)$ be a permutation of 1234. Then

$$W_i = U_{\sigma(i)}$$

$$\begin{aligned} W_1 &= U_{\sigma(1)} = U_2 \\ W_2 &= U_{\sigma(2)} = U_3 \\ &\vdots \end{aligned}$$

Lemma:

$$f_{W_1 \cdots W_n}(w_1', \ldots, w_n') = \begin{cases} n!\, t^{-n} & 0 \le w_i' < t \quad i = 1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$$

Pf: $n=2$. $f_{W_1 W_2}(w_1', w_2')\, \Delta w_1 \Delta w_2$

$$\approx P(w_1' < W_1 < w_1' + \Delta w_1,\ w_2' < W_2 < w_2' + \Delta w_2)$$

$$= P(w_1' < U_1 < w_1' + \Delta w_1,\ w_2' < U_2 < w_2' + \Delta w_2)$$

$$+ P(w_1' < U_1 < w_1' + \Delta w_1,\ w_2' < U_2 < w_2' + \Delta w_2)$$

$$= 2\left(\frac{\Delta w_1}{t}\right)\left(\frac{\Delta w_2}{t}\right) = \frac{2!}{t^2}$$

Thm: Let $W_1, W_2, \ldots$ be waiting times of a Poisson process. Conditioned on $N(t) = n$

$$f_{W_1 \cdots W_n \mid X(t)}(w_1', \ldots, w_n', n)$$

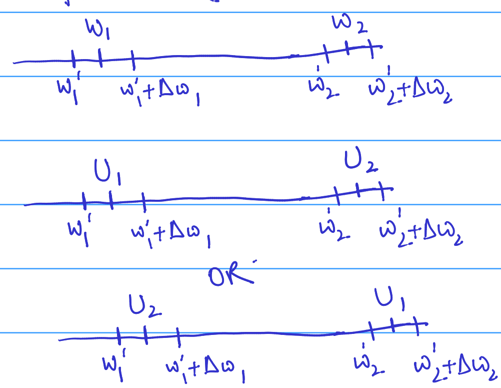
conditional on n points in $[0,t)$

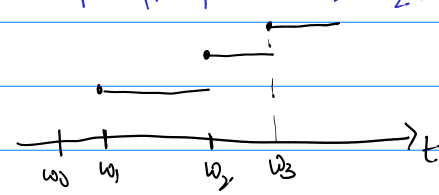
Joint density of $W_1, \ldots, W_n$

$$= n!\, f_{U_1 \cdots U_n}(w_1', \ldots, w_n)$$

$$f_{W_1 \cdots W_n}(w_1', w_2') = \begin{cases} 2! \dfrac{1}{t^2} & 0 \le w_i \le t \\ 0 & \text{otherwise} \end{cases}$$

$$f(w)\Delta w \approx P(w < W < w + \Delta w)$$





$$= \frac{n!}{t^n} \qquad 0 < w_1 \dots < w_n \le t \qquad = n!\, f_{U_1 \dots U_n}(w_1, \dots, w_n)$$

iid Uniform[0,t)

$$f_{W_1 \dots W_n | X(t)} = \frac{f_{W_1 \dots W_n, X(t)}(w_1 \dots w_n, n)}{P(X(t)=n)}$$

$$f_{X|Y}(x,n) = \frac{f_{X,Y}(x,n)}{P_Y(n)}$$

X is continuous

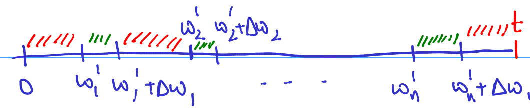
Y is discrete

$$f_{W_1 \dots W_n, X(t)}(w_1', \dots w_n', n)\, \boxed{\Delta w_1 \dots \Delta w_n}$$

Can be defined by this relation.

$$\approx P(w_i' < W_i < w_i' + \Delta w_i, X(t) = n)$$

$$= P(\{\text{No events in } (0, w_1), (w_1 + \Delta w_1, w_2) \dots, (w_n + \Delta_n, t)\} \cap \{\text{exactly one event in } (w_1, w_1 + \Delta w_1), (w_2, w_2 + \Delta w_2) \dots\})$$



Prob (no arrivals in red region $\cap$ 1 arrival in each green region)

$$P(X(w_i', w_i' + \Delta w_i) = 1) = \frac{e^{-\lambda \Delta w_i}(\lambda \Delta w_i)}{1!} = e^{-\lambda \Delta w_i}(\lambda \Delta w_i)$$

$$P(X(w_i' + \Delta w_i, w_{i+1}) = 0) = e^{-\lambda(w_{i+1} - w_i - \Delta w_i)}$$

telescope

$$\rightarrow e^{-\lambda w_1} e^{-\lambda(w_2 - w_1 - \Delta w_1)} \dots e^{-\lambda(t - w_n - \Delta w_n)}$$

$$e^{-\lambda \Delta w_1} e^{-\lambda \Delta w_2} \dots e^{-\lambda \Delta w_n} (\lambda \Delta w_1) \dots (\lambda \Delta w_n)$$

$$= e^{-\lambda t} \lambda^n \Delta w_1 \dots \Delta w_n$$

$$\frac{\cancel{e^{-\lambda t}} \lambda^n \Delta w_1 \dots \Delta w_n}{\frac{\cancel{e^{-\lambda t}} (\lambda t)^n}{n!}}$$

$$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$\Rightarrow f_{W_1 \cdots W_n | X(t)}(w_1' \cdots w_n', n) = \frac{n!}{t^n}$ (this is what we wanted to show).

<u>Lemma</u> Let $f(w_1, \cdots, w_n)$ be such that $f(w_{\sigma(1)} \cdots w_{\sigma(n)}) = f(w_1, \cdots, w_n)$

where $\sigma \in S_n$ is a permutation.

$f(w_1, w_2) = e^{-(w_1 + w_2)}$

$f(w_2, w_1) = e^{-(w_2 + w_1)} = f(w_1, w_2)$

(Invariant under permutations of the variables)

$f(w_1, \cdots w_n) = w_1 w_2 \cdots w_n$

Then

$\mathbb{E}[f(W_1 \cdots W_n) \mid X(t) = n]$

$= \mathbb{E}[f(U_1 \cdots U_n)]$

where $W_1 < W_2 \cdots < W_n$ are the ARRIVALS of the Poisson process.

where U_i are iid Uniform$[0, t)$.

We have some theory about the uniformity of Poisson points.
Will see 3 "real world" applications of this.

Ex: $f(w_1 \cdots w_n) = e^{-\beta w_1} + \cdots + e^{-\beta w_n}]$

$f(w_{\sigma(1)}, \cdots w_{\sigma(n)}) = e^{-\beta w_{\sigma(1)}} + \cdots e^{-\beta w_{\sigma(n)}}$

Pf

$E[f(U_1, \cdots, U_n)]$

n dimensional integral

$$= \int_{\substack{0<u_i<t \\ i=1,\cdots,n}} f(u_1, \ldots, u_n) \frac{du_1 \cdots du_n}{t^n}$$

t^n ↳ joint pdf

Split up this integral into $n!$ different integrals.

$(u_1, u_2, \cdots, u_n) = (0.1, 0.7, 0.2, \cdots)$

$\downarrow$

$(0.1, 0.2, \cdots)$

$(u_2, u_1, u_3, \cdots)$

$$= \int_{0<u_1<u_2\cdots<u_n<t} f(u_1, \ldots, u_n) \frac{du_1 \cdots du_n}{t^n} \quad \text{★1a}$$

$$+ \int_{0<u_2<u_1\cdots<u_n<t} f(u_1, u_2, \ldots, u_n) \frac{du_1 \cdots du_n}{t^n} \quad \text{★1b}$$

Each corresponds to a permutation.

All these integrals are identical.

+ ⋮ (for a total of $n!$ terms)

In ★1b make the COV $u_1' = u_2$, $u_2' = u_1$, $u_3' = u_3, \cdots$

f is invariant under perm.

$$\int_{0<u_1'<u_2'\cdots<u_n<t} f(u_2', u_1', \ldots, u_n') \frac{du_1' \cdots du_n'}{t^n}$$

But we $f(u_1', u_2' \cdots u_n) = f(u_2', u_1', \cdots, u_n')$

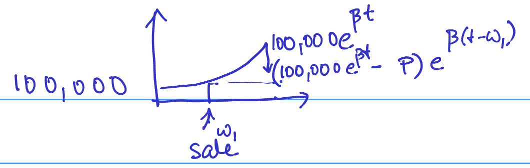
$$= \int_{0<u_1'<u_2'\cdots<u_n<t} f(u_1', u_2', \cdots, u_n') \frac{du_1' \cdots du_n'}{t^n}$$

$=$ ★1a

So $E[f(U_1, \cdots, U_n)] = n! \int_{0<u_1<\cdots<u_n<t} f(u_1 \cdots u_n) \frac{du_1 \cdots du_n}{t^n}$

$= E[f(W_1, \cdots, W_n) \mid X(t) = n]$

Ex: Customers arrive at a car dealership as a Poisson process with rate λ. You borrow 100,000 $(= C)$ amount of dollars from the bank to buy $(10 = N)$ cars from the manufacturer. They've borrowed at some rate of interest (β), and you want to decide how much to price the cars (P_0)

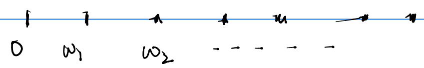


Amount of money each sale is worth is dependent on how soon they come in.

Present value of customer purchases

$$= e^{-\beta W_1} P + e^{-\beta W_2} P + \cdots + e^{-\beta W_k} P$$

So you compute the "present value" of your sales after a year. $(= t)$



$X(t) =$ Poisson, # of arrivals/customers by time t with rate λ.

If you get your first customer at time W_1, you must discount the price of the car P by $e^{-\beta w_1} P$

Then $M(t) =$ "Present Expected value of car sales at time t"

$$= P\,\mathbb{E}\left[\sum_{i=1}^{X(t)} e^{-\beta W_i}\right]$$

cost of car sales

Random sum!

is invariant under permutations

$$= \sum_{n=1}^{\infty} \underbrace{\mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta W_i} \,\middle|\, X(t)=n\right]}_{\text{This sum simplifies since}} \mathbb{P}(X(t)=n)$$

$$\mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta U_i}\right] = \sum_{i=1}^{n} \mathbb{E}\left[e^{-\beta U_i}\right] = n\,\mathbb{E}\left[e^{-\beta U_1}\right]$$

$$f(W_1, \dots, W_n) = e^{-\beta W_1} + \dots + e^{-\beta W_n}$$

is invariant under permutation.

$$= \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta U_i}\right] P(X(t)=n)$$

where $U_i \sim \text{Uniform}[0,t)$ iid

$$= \sum_{n=1}^{\infty} n\,\mathbb{E}\left[e^{-\beta U_1}\right] P(X(t)=n) = \mathbb{E}\left[e^{-\beta U_1}\right] \underbrace{\sum_{n=1}^{\infty} \underbrace{n}_{\text{value}} \underbrace{P(X(t)=n)}_{\text{pmf}}}_{\text{Expectation (Poisson }(\lambda t))}$$

$$= \lambda\,\mathbb{E}\left[e^{-\beta U_1}\right]$$

$$\mathbb{E}\left[e^{-\beta U_1}\right] = \int_0^t \underbrace{e^{-\beta s}}_{\text{value}} \underbrace{\frac{ds}{t}}_{\text{pdf}} = \frac{1-e^{-\beta t}}{\beta t}$$

$$M = P \sum_{n=1}^{\infty} \frac{(1-e^{-\beta t})}{\beta t} n P(X(t)=n)$$

$$= P \lambda \frac{(1-e^{-\beta t})}{\beta t}$$ ← time one year

choose P such that

β known interest rate

λ guess based on Market research.

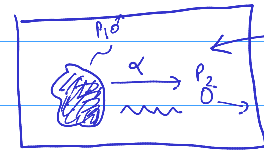
Should plan for $M > C$ (your loan amount)

Suppose you want to account for the fixed amount of cars you have, then

$$M = E\left[\sum_{n=1}^{\min(X(t), N)} e^{-\beta w_i}\right] P$$

☆ Practice computation for exam and/or quiz.

Ex: Arrivals with a lifetime/expiration



# of particles in the room at time t.

"Its known" that

So if you have a lump of Uranium or other radioactive material lying around then it will release α particles (or β particles or γ rays) every once in a random time

The interarrival times at which these decays happen may be modeled as exponential iid waiting times. [The rate is usually measured in terms of the $\boxed{\text{half life}}$] Exp(λ)

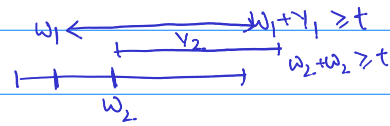
Let $X(t)$ = # of particles created up to time t.

The α particles then decay independently themselves (they die) with a random variable Y_k. → cdf of Y_k is given by a function G.

Let Y_k be the lifetime of the k^{th} particle born at time W_k

I want to find $M(t)$, the # of existing particles at time t. (given $M(0)=0$)

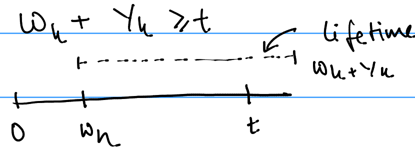
Clearly $\overbrace{M(t)}^{\text{existing particles}} \leq \overbrace{X(t)}^{\text{total \# of particles released}}$.



Condition on there being n particles at time t. $(X(t)=n)$. Ask if

The k^{th} particle at time exists at time t. This would be true if

$W_k + Y_k \geq t$



$$\underbrace{\sum_{k=1}^{n} 1_{\{W_k+Y_k \geq t\}}}_{\tilde{M}(t)} = \text{\# of particles that exist at time } t.$$

prob. of having m particles surviving at time t given n particles have been released

$$P(M(t)=m \mid X(t)=n)$$

$$= P\Big(\underbrace{\sum_{k=1}^{n} \mathbb{1}_{\{\overset{U_k}{W_k}+Y_k \geq t\}}}_{\tilde{M}(t)} = m \;\Big|\; X(t)=n\Big)$$

$$\mathbb{E}[f(W_1 \cdots W_n) \mid X(t)=n] = \mathbb{E}[f(U_1 \cdots U_n)]$$

$$\mathbb{E}[\mathbb{1}_A] = P(A) \qquad A = \Big\{\sum_{k=1}^{n} \mathbb{1}_{\{W_k+Y_k \geq t\}} = m\Big\}$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{\hat{M}(t)=m\}}(W_1, \ldots W_n, Y_1 \ldots Y_n) \;\Big|\; X(t)=n\Big]$$

TOWER PROPERTY

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}\big[\mathbb{1}_{\hat{M}(t)=m} \mid W_1, \cdots, W_n, X(t)=n\big]}_{f(W_1, \ldots, W_n)} \;\Big|\; X(t)=n\Big] \quad ☆3$$

(defined to be my f)

(Easy to see that it's invariant under permutations.)

$$f(W_2, W_1, \cdots, W_n) =$$

I can swap Y_1 and Y_2 since they're iid

$$\mathbb{E}\Big[\mathbb{1}\Big(\mathbb{1}_{\{W_2+Y_1 \geq t\}} + \mathbb{1}_{\{W_1+Y_2 \geq t\}} + \cdots\Big) = m \;\Big|\; W_1 \ldots W_n\Big]$$

$$= f(W_1, \ldots W_n) \quad \text{(since } Y_i \text{ are iid)}$$

Thus ($\star$3) is

Ui are iid Uniform $[0,t)$.

$$= P\left(\sum_{i=1}^{n} 1_{\{U_i + Y_i \geq t\}} = m \right) \quad = \quad P\left(\sum_{i=1}^{n} 1_{\{W_i + Y_i \geq t\}} = m \;\middle|\; X(t) = n \right)$$

$$= P\left(\text{Bin}(n,p) = m \right)$$

$X \sim \text{Bin}(n,p)$ = # of success in n trials.

$$X = 1_{\{T_1 = H\}} + 1_{\{T_2 = H\}} + \cdots + 1_{\{T_n = H\}}$$

where $p = P(U_i + Y_i \geq t)$

$$P(X = m) = P\left(\sum_{i=1}^{n} 1_{\{T_i = H\}} = m \right)$$

$P(T_i = H) = p$

$$\int_0^t P(Y_i \geq t - u \mid U_i = u) \underbrace{\frac{1}{t}}_{\text{pdf} = P(U_i = u)} du$$

$P(Y_i \leq x) = G(x)$

$z = t-u,\ -dz = du$

$$= \frac{1}{t} \int_0^t (1 - G(t-u))\, du = \underbrace{\frac{1}{t} \int_0^t (1 - G(z))\, dz}_{p}$$

p complicated (involves cdf of lifetime)

So finally to find

$$P(M(t) = m) = \sum_{n=m}^{\infty} P(M(t) = m \mid X(t) = n) \times P(X(t) = n)$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= p^m e^{-\lambda t} \sum_{n=m}^{\infty} \frac{\cancel{n!}}{(n-m)!\, m!} \frac{1}{\cancel{n!}} (\lambda t)^{n-m} (\lambda t)^m (1-p)^{n-m}$$

$$= \frac{(\lambda t)^m p^m e^{-\lambda t}}{m!} \sum_{n=m}^{\infty} \frac{[(\lambda t)(1-p)]^{n-m}}{(n-m)!}$$

$i = n-m$

$$= \frac{(\lambda t)^m p^m e^{-\lambda t}}{m!} \underbrace{\sum_{i=0}^{\infty} \frac{(\lambda t(1-p))^i}{i!}}_{\text{Exponential series.}}$$

$$= \frac{(\lambda t)^m p^m e^{-\lambda t}}{m!} e^{\lambda t(1-p)}$$

$$= \boxed{\frac{(\lambda t p)^m e^{-\lambda t p}}{m!}}$$

Poisson!

$= \text{Poisson}(\lambda t p)$ # of particles surviving at time t distribution.

What happens as $t \to \infty$?

$$p = \frac{1}{t}\int_0^t (1 - G(z))\, dz$$

$$\int_0^{\infty} P(Y \geq z)\, dz$$

$$\Rightarrow tp \xrightarrow{t \to \infty} \int_0^{\infty} (1 - G(z))\, dz = E[Y]$$

Which gives you a cool result!!

As $t \to \infty$ the # of surviving particles $\sim$ Poisson $(\lambda E[Y])$

rate of arrival mean lifetime.

It becomes independent of the other features of Z, and only cares about the mean lifetime.

Could you have guessed that?

Sum Quota Sampling

Suppose an airline observes over a period of two years

1) Part fails at 7 months
2) Replacement fails at 5 months
3) " " 9 months

The key is the 2 years period.

Q: What is the avg. lifetime of the parts?

$$= \frac{7+5+9}{3} \;? \qquad \text{Or} \qquad \frac{24}{3} = \frac{\text{Total time elapsed}}{\#\text{ of parts failed}}$$

Let $X_i \sim$ lifetime of part i

If $X_i \sim Exp(\lambda)$ iid

Then $N(t) = \#$ of parts that have elapsed by time t

$\sim$ Poisson Process with parameter λ.

The avg we computed is:

$$\overline{X}_{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)}$$

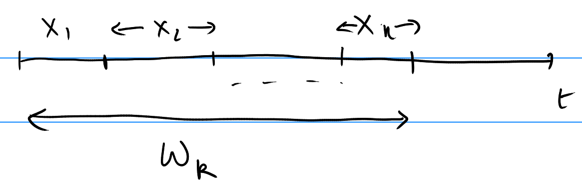
How close is $E[\overline{X}_{N(t)}]$ to $E[X_1]$ the true mean lifetime?

Well it only makes sense to compute

$$E[\overline{X}_{N(t)} \mid N(t) > 0]$$

otherwise we would divide by 0. So let's compute

$$E[\overline{X}_{N(t)} \mid N(t) = k] = E\left[\frac{W_k}{k} \,\Big|\, N(t) = k\right]$$



$$= \frac{E[\max(U_1, \dots, U_k)]}{k} \qquad U_i \sim \text{Uniform}(0,t)$$

$$P(\max(U_1, \dots, U_k) \le s) = \prod_{i=1}^{k} P(U_i \le s)$$

$$= \left(\int_0^s \frac{1}{t}\,dx\right)^k = \frac{s^k}{t^k}$$

$$E[\max(U_1, \dots, U_k)] = \int_0^t 1 - \frac{s^k}{t^k}\,ds = t - \frac{t^{k+1}}{t^k(k+1)}$$

$$= t\left(\frac{k+1-1}{k+1}\right) = \frac{kt}{k+1}$$

$$E[\overline{X}_{N(t)} \mid N(t) > 0] = \frac{E[\overline{X}_{N(t)}, N(t) > 0]}{P(N(t) > 0)}$$

$$= \frac{E[\bar{X}_{N(t)} \mid N(t)=k]\, P(N(t)=k)}{P(N(t)>0)}$$

$$= \sum_{k=1}^{\infty} \frac{t}{k+1} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \Big/ 1-e^{-\lambda t}$$

$$= \frac{t e^{-\lambda t}}{1-e^{-\lambda t}} \frac{1}{\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k+1}}{(k+1)!}$$

$$= \frac{t e^{-\lambda t}}{} \frac{1}{\lambda t} \left(\frac{e^{\lambda t} - 1 - \lambda t}{e^{\lambda t}-1} \right)$$

Compute : $\frac{\text{Difference between computed \& true}}{\text{True mean life}} \times 100$

$=$ % Bias of computed mean life.

$$= \frac{E[\bar{X}_{N(t)}] - E[X_1]}{E[X_1]}$$

$$= \frac{\frac{1}{\lambda}\left(1 - \frac{\lambda t}{e^{\lambda t}-1}\right) - \frac{1}{\lambda}}{1/\lambda} = \frac{\lambda t}{e^{\lambda t}-1}$$

Note that $E[\bar{X}_{N(t)}] \geq (1+\underbrace{\varepsilon}_{+ve}) E[X_1]$

1) So we would estimate a LARGER than normal lifetime.

2) $\dfrac{\lambda t}{e^{\lambda t}-1} \longrightarrow 0$ exponentially fast.

$\lambda t = E[N(t)] =$ Expected # of failures

$E[N(t)]$	Bias
1	0.58
2	0.31
3	0.16
4	.
5	.
6	.
10	0.005

Law of Rare events redux

Let $S_n = X_1 + \cdots + X_n$

$X_i \sim \text{Bernoulli}(p_i)$

$Y \sim \text{Poisson}(\mu) \qquad \mu = \sum_{i=1}^{n} p_i$

$$\left| P(S_n = k) - P(Y = k) \right| \leq \sum_{i=1}^{n} p_i^2$$

Pf: Recall $A_1 \sim \text{Poisson}(\mu_1)$ + $A_2 \sim \text{Poisson}(\mu_2)$

$A_1 + A_2 \sim \text{Poisson}(\mu_1 + \mu_2)$

So $Y_n = A_1 + A_2 + \cdots + A_n \qquad A_i \sim \text{Poisson}(p_i)$

Then
$$S_n = X_1 + \cdots + X_n$$
$$Y_n = A_1 + \cdots + A_n$$

$$P(S_n = k) = P(S_n = k, Y_n = k) + P(S_n = k, Y_n \neq k)$$

$$\leq P(Y_n = k) + P(S_n \neq Y_n)$$

Similarly
$$P(Y_n = k) \leq P(S_n = k) + P(S_n \neq Y_n)$$

$$\Rightarrow \left| P(S_n = k) - P(Y_n = k) \right| \leq P(S_n \neq Y_n)$$

When is $S_n \neq Y_n$? When 1 of the $X_i \neq A_i$

$$\Rightarrow P(S_n \neq Y_n) \leq P\left(\bigcup_{i=1}^{n} X_i \neq A_i\right)$$

$$\leq \sum_{i=1}^{n} P(X_i \neq A_i)$$

$X_i \sim \text{Bernoulli}(p_i)$

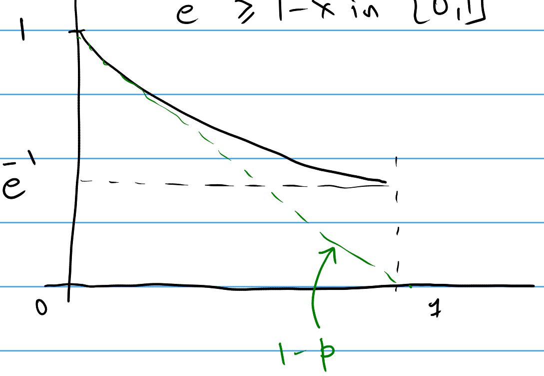
$A_i \sim \text{Poisson}(p_i)$

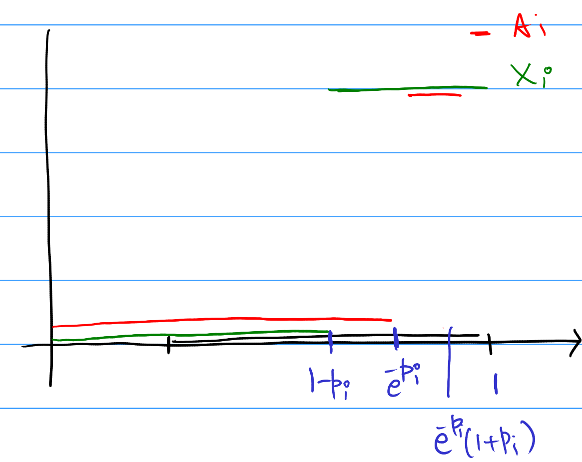
Lets not make X_i and A_i iid, we will couple them!

Let $U_i \sim \text{Uniform}[0,1]$ iid

$$X_i = \begin{cases} 0 & U_i \leq 1-p_i \\ 1 & U_i > 1-p_i \end{cases} \qquad A_i = \begin{cases} 0 & U_i \leq e^{-p_i} \\ 1 & e^{-p_i} < U_i \leq e^{-p_i} p_i + e^{-p_i} \\ \vdots & \\ k & \sum_{j=0}^{k-1} e^{-p_i} p_i^j < U_i \leq \sum_{j=0}^{k} e^{-p_i} p_i^j \end{cases}$$

$e^{-x} \geq 1-x$ in $[0,1]$





$$\Rightarrow P(X_i \neq A_i) = P(1-p_i \leq U_i \leq e^{-p_i}) + P(U_i > e^{-p_i}(1+p_i))$$

$$= \cancel{e^{-p_i}} - \cancel{1} + p_i + \cancel{1} - e^{-p_i}(\cancel{1}+p_i) = p_i - p_i e^{-p_i}$$

$$\leq p_i - p_i(1-p_i) = p_i^2$$